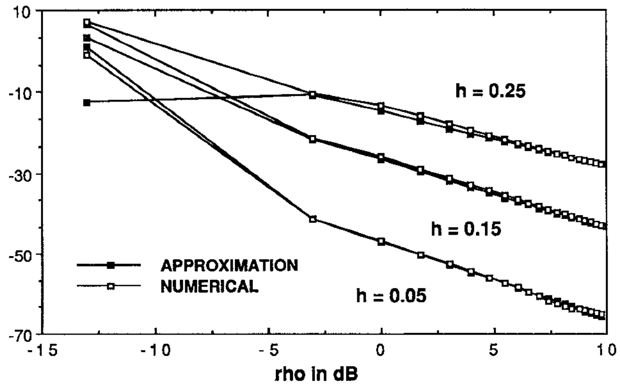

Fig. 2. Comparison of numerical calculations and the analytical approximation for $|V|$. Distance is in dB: $\log_{10} k_0\rho$. Results are shown for $k_0h = 0.05$, 0.15, and 0.25.

and $k_0\rho(20\log_{10}|k_0\rho|)$. The agreement between the numerical calculations and analytical approximation gets progressively worse as the substrate thickness is increased. The largest error occurs for the value of $k_0\rho$ near -13 dB. The reason is that the approximation for V in (19) has two $1/\rho$ terms in it. These terms dominate all others for small values of ρ. The first of these terms is due to the approximate expression for $\rho \gg h$. The second is due to the direct term in the quasi-static approximation. The direct term should dominate for distances close to the dipole, as can be seen from the numerical results. The first $1/\rho$ term is assumed to be negligible compared with the direct term, as it is smaller by a factor $|k_0h|^2$. It is seen from (19) that this is true when

$$|kh_0|^2 \ll \frac{\epsilon_r^2}{(\epsilon_r - 1)^2(\epsilon_r + 1)}. \tag{22}$$

This condition is violated for values of $k_0h = 0.25$. The solution to this problem is that one should not include the first $1/\rho$ term for small values of ρ. (The surface wave term should not be included either. This, however, gives negligible corrections, as the surface wave will go as $\log|\rho|$ for small values of ρ, which is much smaller than $1/\rho$.)

The maximum relative error for the points plotted for $k_0h = 0.05$ is about 1.5% if the $1/\rho$ term mentioned above is left out for $\rho < h$. Similarly, the maximum relative error for $k_0h = 0.15$ is approximately 6%, and for $k_0h = 0.25$ is approximately 11% for the points plotted.

ACKNOWLEDGMENT

The author wishes to thank B. Brim, who developed the code to numerically evaluate the exact integrals.

REFERENCES

[1] J. R. Mosig, and F. E. Gardiol, "Analytical and numerical techniques in the Green's function treatment of microstrip antennas and scatterers," *Proc. Inst. Elec. Eng.*, vol. 130, pt. H, no. 2, pp. 175–182, 1983.
[2] J. R. Mosig, and F. E. Gardiol. "General integral equation for microstrip antennas and scatterers," *Proc. Inst. Elec. Eng.*, vol. 132, pt. H, no. 7, pp. 424–432, Dec. 1985.
[3] J. Mosig, "Integral equation technique," in *Numerical Techniques for Microwave and Millimeter-Wave Passive Structures*, by T. Itoh, Ed. New York: Wiley, 1989, ch. 4.
[4] M. Marin, S. Barkeshli, and P. H. Pathak, "Efficient analysis of planar microstrip geometries using a closed-form asymptotic representation of the grounded dielectric slab Green's function," *IEEE Trans. Microwave Theory Tech.*, vol. 37, pp. 669–679, Apr. 1989.
[5] L. B. Felsen and N. Marcuvitz, *Radiation and Scattering of Waves*. Englewood Cliffs, NJ: Prentice-Hall, 1973.
[6] T. T. Wu and R. W. King, "Lateral waves: a new formula and interference patterns," *Radio Sci.*, vol. 17, no. 3, pp. 521–531, 1982.
[7] T.T. Wu and R. W. P. King, "Lateral waves: New formulas for $E_{1\phi}$ and E_{1z}," *Radio Sci.*, vol. 17, no. 3, pp. 532–538, 1982.
[8] R. W. P. King, "The propagation of signals along a three layered region: Microstrip," *IEEE Trans. Microwave Theory Tech.*, vol. 36, pp. 1080–1086, June 1988.
[9] M. Abramowitz and I. A. Stegun, *Handbook of Mathematical Functions* (AMS 55). Washington, DC: National Bureau of Standards, 1972.

Efficient Computation of the Free-Space Periodic Green's Function

Surendra Singh and Ritu Singh

Abstract —**The application of Shanks's transform is shown to improve the convergence of the series representing the doubly infinite free-space periodic Green's function. Higher order Shanks transforms are computed via Wynn's ϵ algorithm. Numerical results confirm that a dramatic improvement in the convergence rate is obtained for the "on-plane" case, in which the series converges extremely slowly. In certain instances, the computation time can be reduced by as much as a factor of a few thousands. A relative error measure versus the number of terms taken in the series is plotted for various values of a convergence factor as the observation point is varied within a unit cell. Computation times are also provided.**

I. Introduction

The problem of determining the radiation or scattering from a periodic array geometry is formulated in terms of an integral equation. The integral equation is solved numerically via the method of moments. In the moment method solution the unknown surface current or field is expanded either in terms of entire domain basis functions at the expense of generality or in terms of subdomain basis functions at the expense of higher computation cost. In order to achieve the degree of generality required in developing general-purpose computer codes, it is necessary to employ subsectionally defined basis functions. This requires repeated computations of the free-space periodic Green's function. The Green's function for a two-dimensional periodic array (of point sources of radiating elements or conducting strips) is represented in terms of a doubly infinite series. This series converges extremely slowly as the observation point approaches the source plane. In the moment method solution for the current distribution on the radiator in the reference cell of a two-dimensional infinite periodic array of radiating elements, the observation point lies in the plane of the array. This case is referred to as the on-plane case, and the series has the slowest convergence rate. In comparison with other methods that make use of Kummer's transform to accelerate the conver-

Manuscript received October 23, 1990; revised February 14, 1991.
The authors are with the Department of Electrical Engineering, University of Tulsa, Tulsa, OK 74104.
IEEE Log Number 9100146.

gence of the doubly infinite series and thus add complexity as well as evaluation of additional terms, the Shanks transform is simple and efficient.

Methods to enhance the convergence of the periodic Green's function series, which make use of the spectral- and spatial-domain formulations in conjunction with Kummer's and Poisson's transformations, have been suggested [1]–[6]. It has been shown in [7] that an application of Shanks's transform [8] to the series representing the Green's function for a one-dimensional array of point sources accelerates the convergence of the series. In this work, it is shown that the application of Wynn's ϵ algorithm [9], which implements higher orders of Shanks's transform, improves the convergence rate of the series representing the free-space periodic Green's function for a two-dimensional array of point sources.

II. Free Space Periodic Green's Function

The free-space periodic Green's function is given by [10]

$$G(\boldsymbol{r}) = \sum_{m=-\infty}^{+\infty} \sum_{n=-\infty}^{+\infty} \frac{1}{j2Ak_{zmn}} e^{-jk_{zmn}|z|} e^{-j\boldsymbol{k}_{tmn}\cdot \boldsymbol{r}} \tag{1}$$

where $\boldsymbol{r}$ is the location of the observation point and it is assumed without the loss of generality that the reference source is at the origin. Also, A is the area of the unit cell and

$$k_{zmn} = \begin{cases} \sqrt{k^2 - |\boldsymbol{k}_{tmn}|^2}, & k < |\boldsymbol{k}_{tmn}| \\ -j\sqrt{|\boldsymbol{k}_{tmn}|^2 - k^2}, & k < |\boldsymbol{k}_{tmn}| \end{cases} \tag{2}$$

$$\boldsymbol{k}_{tmn} = (m + m_0)\boldsymbol{k}_1 + (n + n_0)\boldsymbol{k}_2 \tag{3}$$

where

$$\boldsymbol{k}_1 = \left(\frac{2\pi}{D_x}\right)\hat{x} \qquad \boldsymbol{k}_2 = \left(\frac{2\pi}{D_y}\right)\hat{y}. \tag{4}$$

Here $\boldsymbol{k}_1$ and $\boldsymbol{k}_2$ are the reciprocal lattice base vectors, defined in (4) for a rectangular lattice; D_x and D_y are the periodicities in the x and y directions respectively; m_0 and n_0 are the interelement phase shift constants; and k is the free-space wavenumber. The series in (1) converges rapidly whenever $z \neq 0$, which is the "off-plane" case, in which the exponential factor aids in the convergence. As the observation point approaches the plane of the array, i.e., as $z \to 0$, the series in (1) converges extremely slowly. Because of this unattractive feature of the Green's function series the use of subdomain basis functions in a moment method solution becomes computationally expensive. In this work, we make use of Shanks's transform to accelerate the summation of this series.

In the application of Shanks's transform to the double summation in (1), the transform is first applied to the sequence of inner partial sums over index n, for a specific value of index m, to arrive at a convergent sum S_m. After obtaining the minimum number of these outer partial sums, the transform is applied to this sequence as well. This process of applying the transform successively to the inner partial sum (over index n) and the outer partial sums (over index m) is continued until a predefined convergence criterion is satisfied.

III. Numerical Results

In this section, we present numerical results on the convergence of the series in (1) with and without the application of Shanks's transform. A straightforward summation of the series

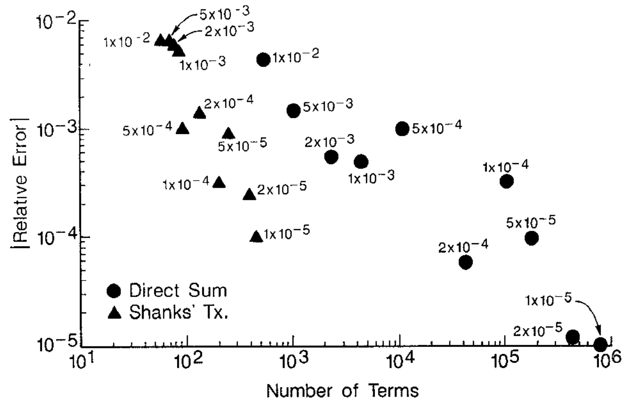

Fig. 1. Relative error magnitude versus the number of terms in the series in (1) for the source point at the origin and the observation point at $(x, y, z) = (0.8\lambda, 0.8\lambda, 0.0\lambda)$.

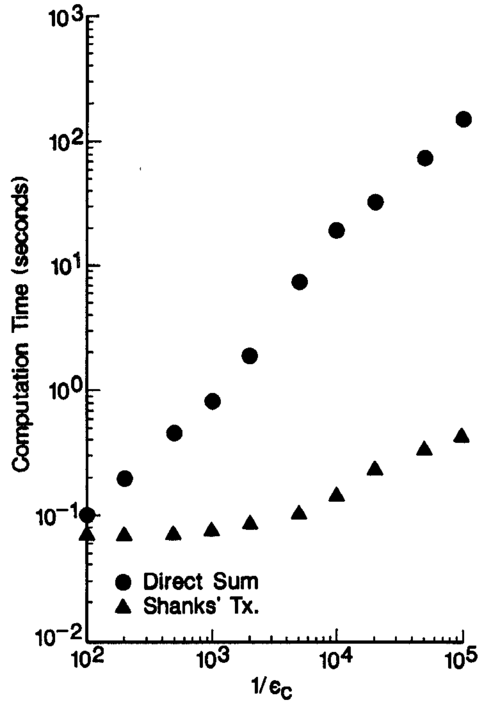

Fig. 2. Computation time in seconds versus $1/\epsilon_c$ for the case in Fig. 1.

is referred to as direct sum. In arriving at the final result for the direct sum and that using the transform, the summation process is terminated when a predefined convergence factor, ϵ_c, defined in [2], is satisfied. Without loss of generality, the source point is taken to be at the origin and the observation point is taken at different locations in the unit cell. For each case, the series in (1) is first summed to machine precision. The resulting sum is then employed in computing a relative error measure for different values of ϵ_c [2]. The following parameters are taken for the numerical results in Figs. 1–8: $D_x = D_y = 1.2\lambda$, $m_0 = n_0 = 0$, and $\lambda = 1$ m. Fig. 1 shows the relative error versus the number of terms for $(x, y, z) = (0.8\lambda, 0.8\lambda, 0.0\lambda)$. The convergence factor is indicated alongside each point. For $\epsilon_c = 10^{-4}$, the Shanks transform converges in 200 terms whereas the direct sum takes more than 100000 terms.

The computation time versus $1/\epsilon_c$ for this case is shown in Fig. 2. At this point, we define a saving factor, which is the ratio

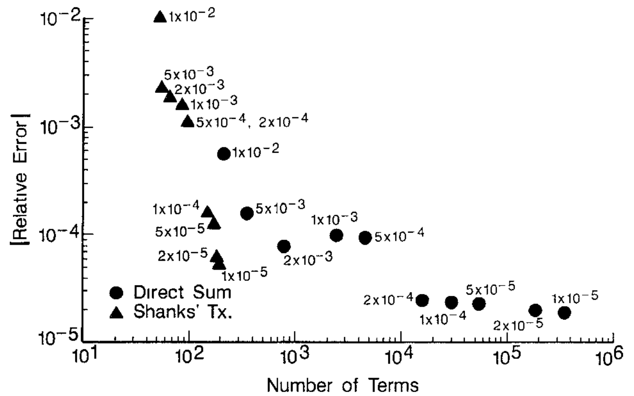

Fig. 3. Relative error magnitude versus the number of terms in the series in (1) for the source point at the origin and the observation point at $(x, y, z) = (0.6\lambda, 0.6\lambda, 0.0\lambda)$.

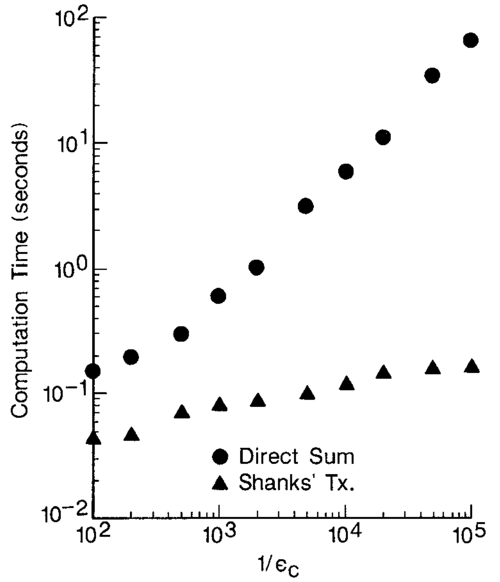

Fig. 4. Computation time in seconds versus $1/\epsilon_c$ for the case in Fig. 3.

of the time taken by the direct sum to that taken by Shanks's transform. For $\epsilon_c = 10^{-4}$ and 10^{-5}, saving factors of 130 and 350, respectively, are obtained. For $(x, y, z) = (0.6\lambda, 0.6\lambda, 0.0\lambda)$ the relative error versus the number of terms and the computation time versus $1/\epsilon_c$ are shown in Figs. 3 and 4, respectively. For $\epsilon_c = 10^{-5}$, Shanks's transform converges in 180 terms and takes 0.15 s while the direct sum takes 350000 terms and takes 68 s. This results in a saving factor of 453.

The series in (1) converges much slower as the observation point is taken closer to the source point at the origin. Next, we take $(x, y, z) = (0.4\lambda, 0.4\lambda, 0.0\lambda)$. The relative error and the computational time for this case are shown in Figs. 5 and 6, respectively. For $\epsilon_c = 10^{-5}$, the direct sum converges in 750000 terms while the Shanks transform converges in 365 terms. The most dramatic results are obtained as the observation point is moved closer to the source point. We take $(x, y, z) = (0.1\lambda, 0.1\lambda, 0.0\lambda)$. Figs. 7 and 8 show the relative error versus the number of terms and the computation time versus $1/\epsilon_c$, respectively. For $\epsilon_c = 2 \times 10^{-5}$, the Shanks transform converges in 800 terms and takes 0.9 s. The direct sum does not converge properly, as shown by the fluctuation in the relative error. However, the prespecified convergence criterion is met in 8.65 million

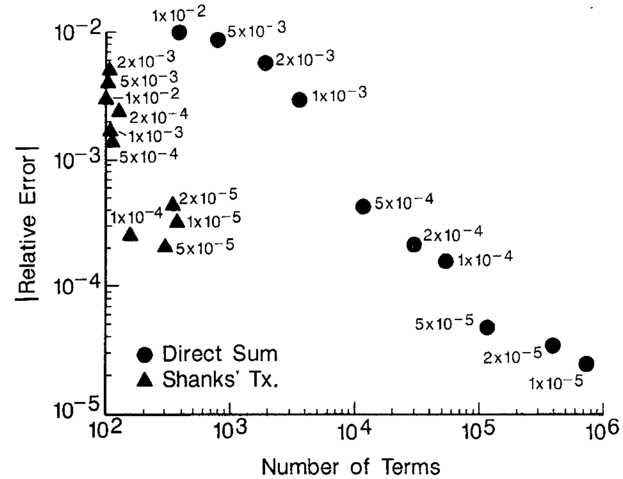

Fig. 5. Relative error magnitude versus the number of terms in the series in (1) for the source point at the origin and the observation point at $(x, y, z) = (0.4\lambda, 0.4\lambda, 0.0\lambda)$.

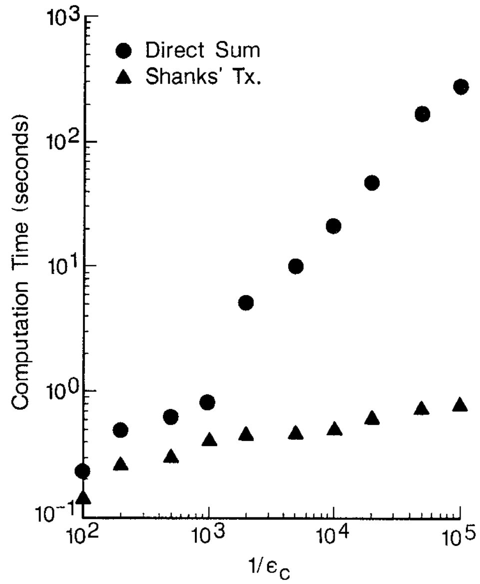

Fig. 6. Computation time in seconds versus $1/\epsilon_c$ for the case in Fig. 5.

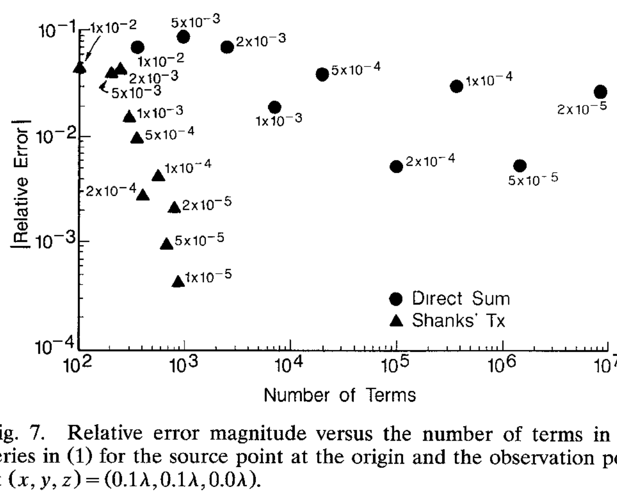

Fig. 7. Relative error magnitude versus the number of terms in the series in (1) for the source point at the origin and the observation point at $(x, y, z) = (0.1\lambda, 0.1\lambda, 0.0\lambda)$.

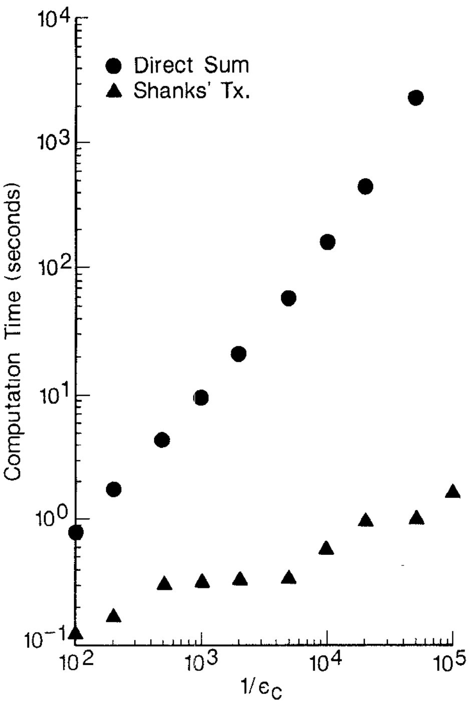

Fig. 8. Computation time in seconds versus $1/\epsilon_c$ for the case in Fig. 7.

terms and takes 2400 s to compute. The saving factor for this case is 2667.

IV. Conclusion

The series representing the free-space periodic Green's function has been accelerated by a simple application of Shanks's transform. Higher order transforms are easily computed via Wynn's ϵ algorithm. It has been shown that the computation time can be reduced by a factor of a few hundreds and, in some instances, a few thousands. This is a significant reduction in computation time as the Green's function is evaluated repeatedly in a moment method solution. The transform is very simple to implement and is extremely efficient, as shown by the numerical results.

References

[1] S. Singh, W. F. Richards, J. R. Zinecker, and D. R. Wilton, "Accelerating the convergence of series representing the free space periodic Green's function," *IEEE Trans. Antennas Propagat.*, vol. 38, pp. 1958–1962, Dec. 1990.

[2] S. Singh and R. Singh, "Application of transforms to accelerate the summation of periodic free- space Green's functions," *IEEE Trans. Microwave Theory Tech.*, vol. 38, pp. 1746–1748, Nov. 1990.

[3] R. Lampe, P. Klock, and P. Mayes, "Integral transforms useful for the accelerated summation of periodic, free-space Green's functions," *IEEE Trans. Microwave Theory Tech.*, vol. MTT-33, pp. 734–736, Aug. 1985.

[4] R. E. Jorgenson and R. Mittra, "Efficient calculation of the free-space periodic Green's function," *IEEE Trans. Antennas Propagat.*, vol. 38, pp. 633–642, May 1990.

[5] R. E. Jorgenson and R. Mittra, "Oblique scattering from lossy strip structures with one- dimensional periodicity," *IEEE Trans. Antennas Propagat.*, vol. 38, pp. 212–219, Feb. 1990.

[6] W. F. Richards, K. McInturff, and P. S. Simon, "An efficient technique for computing the potential Green's function for a thin, periodically excited parallel plate waveguide bounded by electric and magnetic walls," *IEEE Trans. Microwave Theory Tech.*, vol. MTT-35, pp. 276–281, Mar. 1987.

[7] S. Singh and R. Singh, "On the use of Shanks's transform to accelerate the summation of slowly converging series," *IEEE Trans. Microwave Theory Tech.*, vol. 39, pp. 608–610, Mar. 1990.

[8] D. Shanks, "Non-linear transformations of divergent and slowly convergent sequences," *J. Math. Phys.*, vol. 34, pp. 1–42, 1955.

[9] P. Wynn, "On a device for computing the $e_m(S_n)$ transformation," *Math. Tables Other Aids Comput.*, vol. 10, pp. 91–96, 1956.

[10] B. A. Munk and G. A. Burrell, "Plane-wave expansion of arrays of arbitrarily oriented piecewise linear elements and its application to determining the impedance of a single linear antenna in a lossy half-space," *IEEE Trans. Antennas Propagat.*, vol. AP-27, pp. 331–343, May 1979.

Capacitance of a Circular Symmetric Model of a Via Hole Including Finite Ground Plane Thickness

Peter Kok and Daniël De Zutter

Abstract—**The capacitance of a simplified model of a via hole is calculated based on an integral equation approach for the surface charge density. The finite ground plane thickness is explicitly taken into account. Numerical data are obtained for a large range of realistic geometrical data. The relative importance of the contribution to the total capacitance coming from the ground plane opening is explicitly evaluated. It is found that the via capacitance is proportional to the square root of its height, at least for the range of geometrical data considered in this paper.**

I. Introduction

Microstrips and striplines in printed circuit board (PCB) technology for high-frequency/high-speed controlled impedance transport of signals have been extensively studied and modeled [1]. This is much less the case for printed wire technology such as Multiwire® or Microwire® [2], [3]. The parasitic effects caused by discontinuities present in both technologies, such as line crossings, pads, lands, and via holes, form a quite important and still relatively new research topic [4], [5].

In this paper attention is focused on the capacitance of via holes. Via holes provide the connection between lines located in different layers of a multilayered board and therefore have to cross at least one ground plane. Measurements clearly indicate that the effect of realistic via holes is mainly capacitive.

Earlier publications [6], [7] calculate the capacitance and inductance of vias between two different lines above the same ground plane. In [8], capacitance and inductance are calculated for a via hole crossing an infinitely thin ground plane. In this paper, the capacitance of a via hole crossing a ground plane with finite thickness is calculated. To simplify the analysis we have neglected the lines connected by the via.

The formulation of the problem is based on an integral equation for the surface charges combined with an analytical solution at the ground plane opening. The behavior of the via hole capacitance is explicitly studied in terms of the geometrical

Manuscript received July 16, 1990; revised February 11, 1991.

The authors are with the Laboratory for Electromagnetism and Acoustics, University of Ghent, Sint Pietersnieuwstraat 41, 9000 Ghent, Belgium.

IEEE Log Number 9100142.

0018-9480/91/0700-1229$01.00 ©1991 IEEE